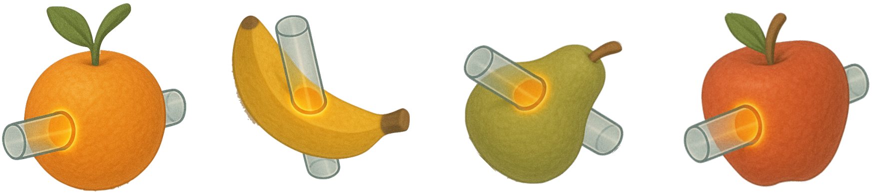
Distance-from-flat persistent homology transforms:

Shoving tubes through shapes gives a sufficient and efficient shape descriptor

Adam Onus, Nina Otter, Renata Turkeš



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$\mathrm{PHT}_{\mathbb{AG}(m,n),\mathrm{d}}$, truncated to homological degrees $\{0, 1, \ldots, m-1\}$ is injective, i.e.
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Motivation: Can PH detect convexity?

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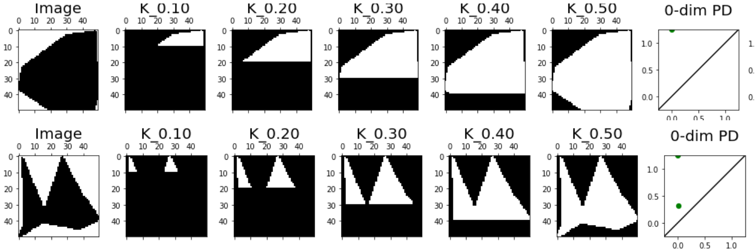

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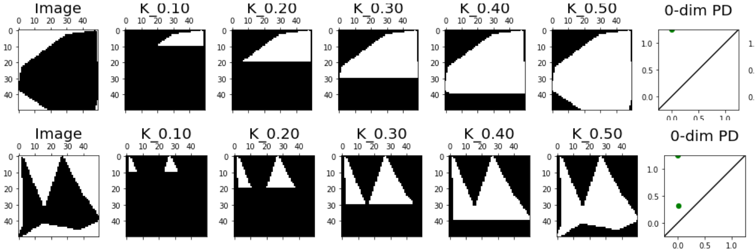

Idea

0-dim PH (connected components) wrt height filtration:

$= 1$ connected component $\Rightarrow$ convex

> 1 connected component $\Rightarrow$ concave

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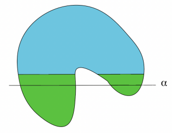

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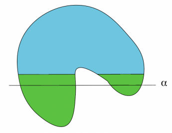

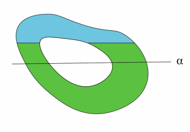

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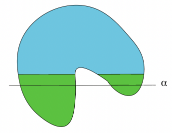

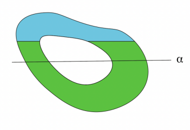

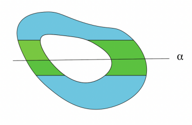

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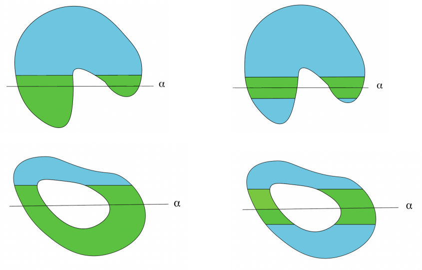

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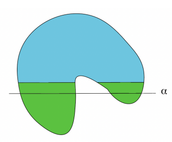

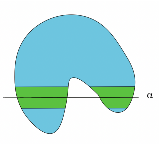

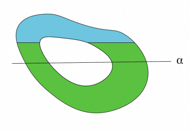

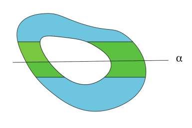

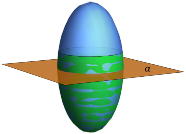

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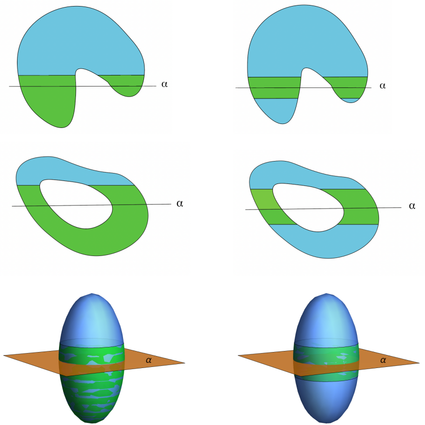

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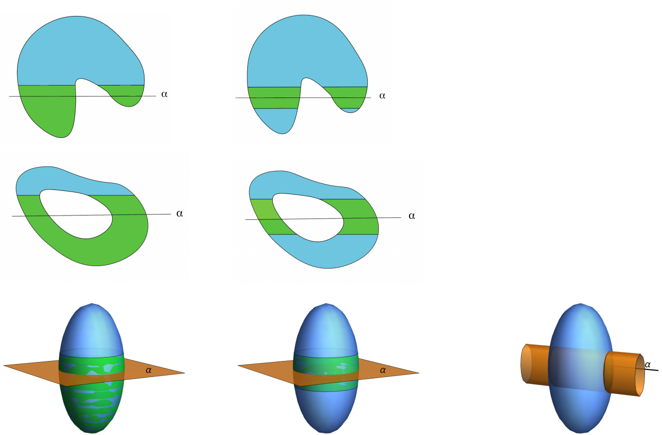

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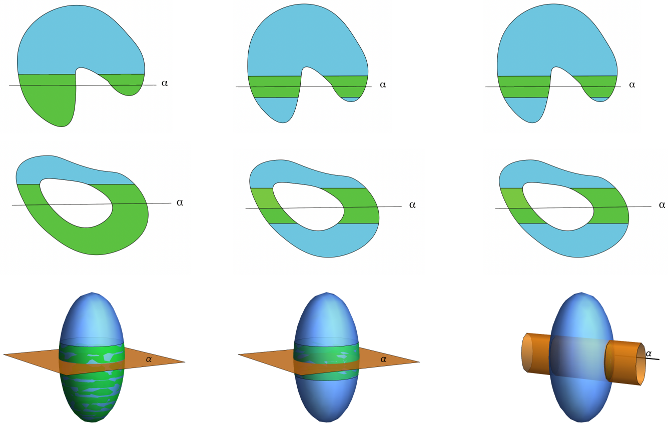

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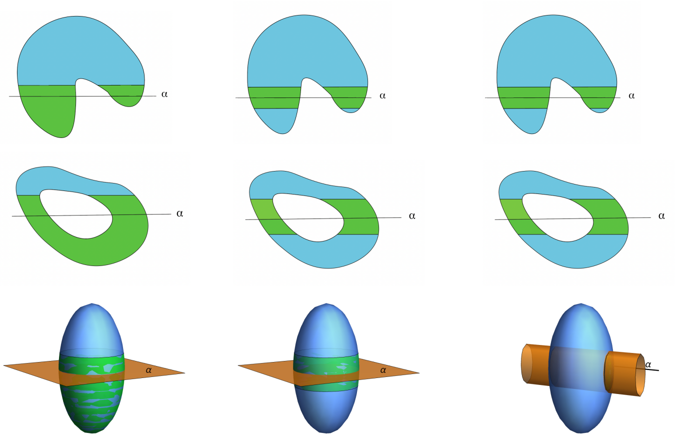

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scalar product
$\eta_v(x) = x \cdot v$

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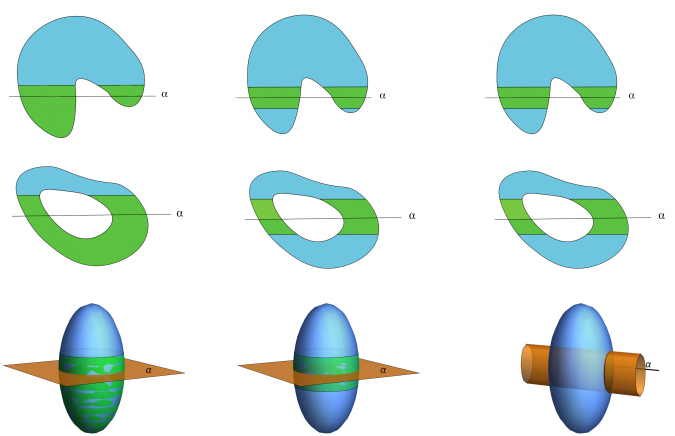

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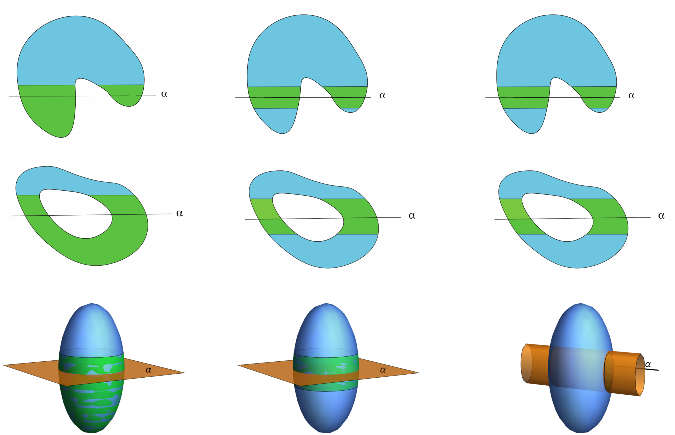
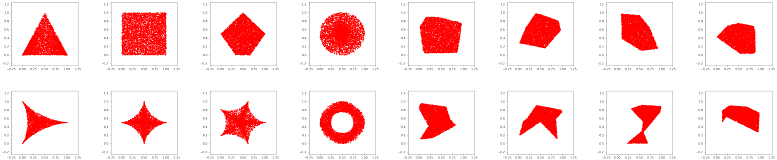


height	absolute height	tubular
scalar product	distance from hyperplane	distance from line
$\eta_v(x) = x \cdot v$	$\eta'_v(x) = \lvert x \cdot v \rvert = d(x, \alpha)$	$\tau_\alpha(x) = d(x, \alpha)$

RT, Guido Montufar and Nina Otter, On the effectiveness of persistent homology, NeurIPS (2022)

Intermezzo: Tubular PHT outperforms NNs



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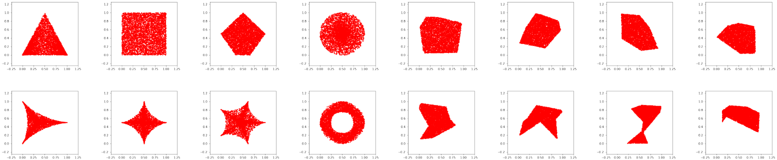

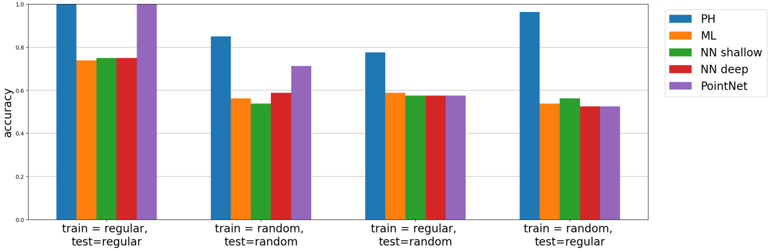

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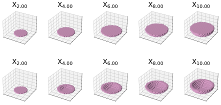
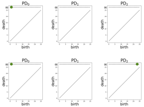
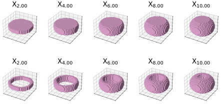
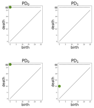
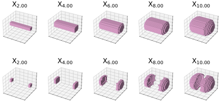
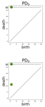
classical (height)		
slabbed (distance-from-hyperplane)		
tubular (distance-from-line)		

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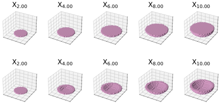
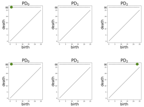
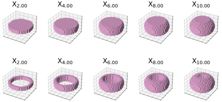
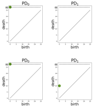
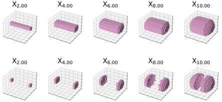
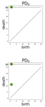
classical (height)		
slabbed (distance-from-hyperplane)		
tubular (distance-from-line)		

Tubular, distance-from-line $\mathrm{PHT}_{\mathbb{AG}(1,n),\mathrm{d}}|_0$ that shoves tubes through shapes is a sufficient and efficient shape descriptor, but one can consider distance from any m-dimensional flat.

Summary

m	Name	$\mathbb{P}$	f_P	k	$\dim(\mathbb{P})$
-	height PHT	$\mathbb{S}^{n-1}$ = sphere in $\mathbb{R}^n$	h_v	$0, 1, \ldots, n-1$	$n-1$
$n-1$	slabbed PHT	$\mathbb{AG}(n-1, n)$ = hyperplanes in $\mathbb{R}^n$	d_P	$0, 1, \ldots, n-2$	n
...	...	...	...	...	...
1	tubular PHT	$\mathbb{AG}(1, n)$ = lines in $\mathbb{R}^n$	d_P	0	$2(n-1)$
0	radial PHT	$\mathbb{AG}(0, n)$ = points in $\mathbb{R}^n$	d_P	"-1"	n

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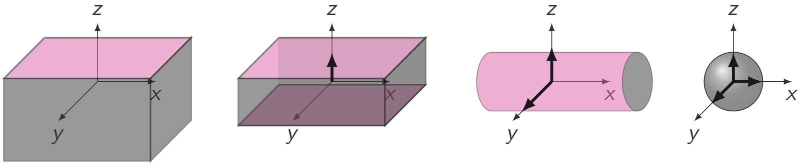

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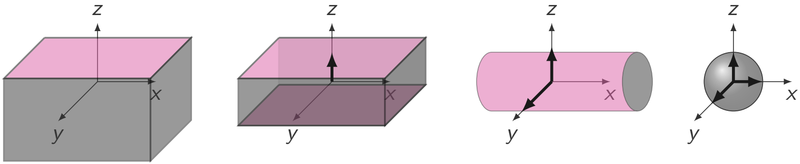

The distance-from-flat $\mathrm{PHT}_{\mathbb{AG}(m,n),\mathrm{d}}$ has a computational advantage over the classical, height $\mathrm{PHT}_{\mathbb{S}^{n-1},h}$, as it is sufficient to calculate less homological degrees k to completely capture the shape; this is more pronounced for lower m.

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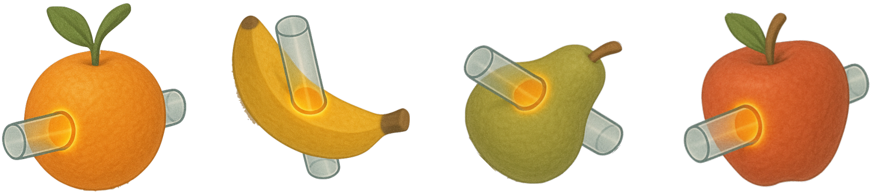

$\mathrm{PHT}_{\mathbb{AG}(m,n),\mathrm{d}}$ is injective

Theorem (Injectivity of $\mathrm{PHT}_{\mathbb{P},f}$)

Let $X \subset \mathbb{R}^n$ be a constructible set, $\mathbb{P}$ a definable set, and $f_P : X \to \mathbb{R}$ a definable function for every $P \in \mathbb{P}$. If

(i) *Euler characteristic of r-level sets, where r takes any value in some $R \subset \mathbb{R}$, completely describes the shape, i.e., the following map is injective*

$$\begin{aligned} \mathcal{CS}(\mathbb{R}^n) &\to \mathcal{CF}(\mathbb{P} \times R) \\ X &\mapsto \big((P, r) \mapsto \chi(X_r(f_P))\big), \end{aligned}$$

(ii) *for every $r \in R$, the Euler characteristic of the r-level set $X_r(f_P)$ can be computed from Euler characteristics of sublevel sets in $\mathbb{R}^m$,*

then $\mathrm{PHT}_{\mathbb{P},f}$, truncated to homological degrees $k \in \{0, 1, \dots, m-1\}$, is injective.

$\mathrm{PHT}_{\mathbb{AG}(m,n),\mathrm{d}}$ is injective: Proof outline

$$
\begin{aligned}
&\mathrm{PHT}_{\mathbb{P},f}(X)\\
&\overset{\mathrm{PHT}}{\Rightarrow}\ (\forall P \in \mathbb{P})(\forall k \in \{0,\dots,n-1\}) : \mathrm{PD}_k(X, f_P)\\
&\overset{\mathrm{PD}}{\Rightarrow}\ (\forall P \in \mathbb{P})(\forall k \in \{0,\dots,n-1\})(\forall r \in R) : \beta_k(X_r^-(f_P))\\
&\overset{\chi}{\Rightarrow}\ (\forall P \in \mathbb{P})(\forall r \in R) : \chi(X_r^-(f_P))\\
&\overset{(ii)}{\Rightarrow}\ (\forall P \in \mathbb{P})(\forall r \in R) : \chi(X_r(f_P))\\
&\overset{(i)}{\Rightarrow}\ X.
\end{aligned}
$$

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(i) $\chi(X \cap P)$ completely describes the shape, due to Schapira's inversion formula for Radon transform,

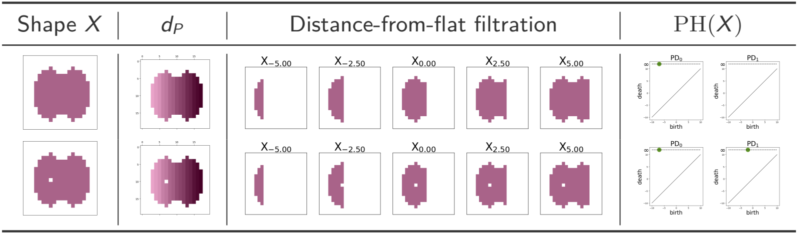
(ii) $\chi(X \cap P)$ can be calculated from PH, since zero-level sets of d_P are zero-sublevel sets.

$\mathrm{PHT}_{\mathbb{AG}(m,n),\mathrm{d}}(X)$ is continuous

$\mathrm{PHT}_{\mathbb{P},f}$	$\mathrm{PHT}_{\mathbb{S}^{n-1},h}$	$\mathrm{PHT}_{\mathbb{AG}(m,n),\mathrm{d}}$
$f_P : X \to \mathbb{R}$ is (Lipschitz) continuous	(✓) ✓	(✓) ✓
$f : \mathbb{P} \to \mathcal{C}(X, \mathbb{R})$ is (Lipschitz) continuous	(✓) ✓	(✗) ✓
$\mathrm{PHT}(X) : \mathbb{P} \to \mathcal{D}^n$ is (Lipschitz) continuous	(✓) ✓	(✗) ✓
$\mathrm{PHT} : \mathcal{CS}(X) \to \mathcal{C}(\mathbb{P}, \mathcal{D}^n)$ is (Lipschitz) continuous	(✗) ✗	(✗) ✗

One can think of four "levels" of continuity within the context of PHT. The continuity of f_P and f together imply the continuity of $\mathrm{PHT}(X)$. The Lipschitz continuity of f ensures the Lipschitz continuity of $\mathrm{PHT}(X)$.

$\mathrm{PHT}_{\mathbb{AG}(m,n),\mathrm{d}}$ is not continuous



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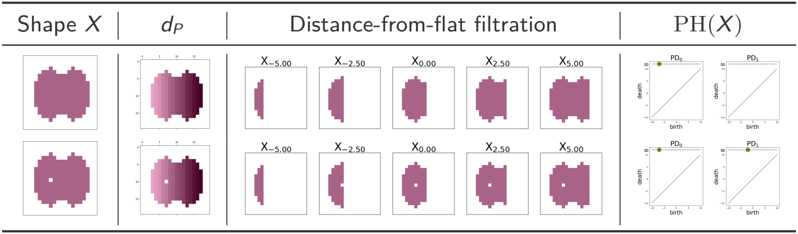

A small shape perturbation can yield a large change in PHT:

$$d(X, X') < \varepsilon,$$

$$W_p(\mathrm{PD}_1(X, d_P), \mathrm{PD}_1(X', d_P)) = \infty.$$

The peculiar case of radial $\mathrm{PHT}_{\mathbb{AG}(0,n),\mathrm{d}}$

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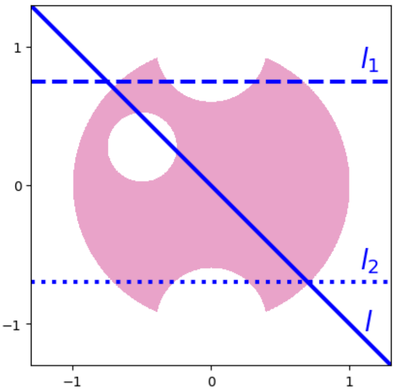
When $m = 0$, it is redundant to calculate PH, since $X \cap P$ is a singleton or an empty set, hence $\chi(X \cap P) = 1$ or $\chi(X \cap P) = 0$.

$\mathrm{PHT}_{\mathbb{G}(m,n),d}$ is not injective



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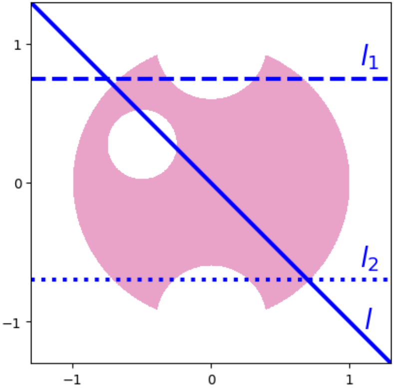
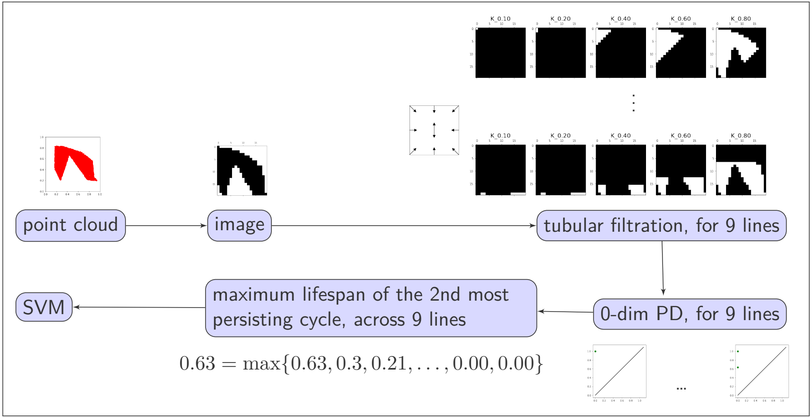


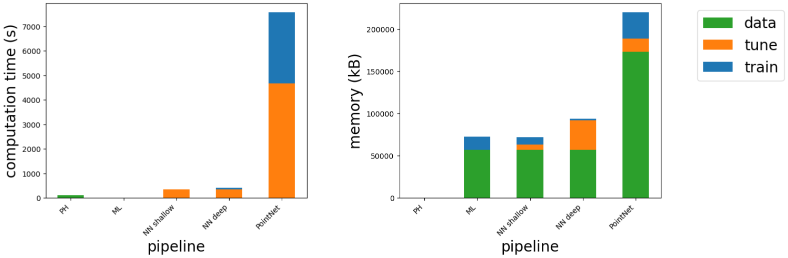
For the example shape, the lines that pass through the origin (such as the line $P = \ell \in \mathbb{G}(1,2)$) can recover the loops, but even after realignment or re-centering, we would need at least one affine line (such as the dashed line $= \ell_1 \in \mathbb{AG}(1,2)$ or the dotted line $= \ell_2 \in \mathbb{AG}(1,2)$) to recover the two dents.

Convexity detection: $\mathrm{PHT}_{\mathbb{AG}(1,n),\mathrm{d}|0}$ pipeline



RT, Guido Montufar and Nina Otter, On the effectiveness of persistent homology, NeurIPS (2022)

Convexity detection: Computational resources



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